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Solution of the elastic boundary value problems for a layer with tunnel stress raisers

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Abstract

A novel procedure for solving three-dimensional problems for elastic layer weakened by through-thickness tunnel cracks has been developed and is presented in this paper. This procedure reduces the given boundary value problem to an infinite system of one-dimensional singular integral equations and is based on a system of homogeneous solutions for a layer. Integral representations of single- and double-layer potentials are used for metaharmonic and harmonic functions entering in the singular integral equations. These representations provide a continuous extendibility of the stress vector while allowing a jump in the displacement vector in the transition through the cut.

Expanding the potential and biharmonic solutions in the Fourier series over the thickness coordinate yields the integral representations of the displacement vector and stress tensor. The problem of reducing a denumerable set of the integral equations of the given boundary value problem to one-to-one correspondence with the set of unknown densities appearing in the Fourier's coefficient representations has been settled efficiently. Numerical investigations show a rapid convergence of the proposed reduction procedure as applied to the solution of the infinite system of one-dimensional integral equations. Numerical examples illustrate the proposed method and demonstrate its advantages.

Keywords: Elastic; Boundary value; Stress; Displacement

1. Introduction

Significant progress in the classical problems of the theory of elasticity for a finite cylinder in R^3 has been made in the last few years because of: more accurate stress analysis of machines, structures, etc., the assessment of their strength and durability regarding various stress raisers. The efficient homogeneous solutions method (HSM), developed by Lur'ye (1942), for three-dimensional problems involving a cylinder (thick plate), was generally employed for stress analysis in circular cylinders. However, it can be extended to other problems in elasticity. In this paper, the HSM is applied to more complicated cases, particularly to a

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layer weakened by a finite number of through tunnel cuts or cavities. Notice that another approach, the eigen-vector functions method was employed by Grinchenko and Ulitko (1970) for the solution of Kirsch's problem for a layer.

In recent years, the finite element method (FEM) has generally been the most employed method for stress analysis in cylinders with cracks. In particular, FEM was applied to the solution of thick plates in tension weakened by a through-thickness rectangular crack (Yamamoto and Sumi, 1978). An interesting interpretation of the HSM in terms of the FEM as applied to the bending deformation of a thick plate with a through-thickness rectangular crack was given by Sundara Raja Iyengar et al. (1988). This reference reviews some investigations related to the above-mentioned problem. Some asymptotic versions of the HSM were given for the solution of multiply connected boundary value problems by Aksentyan and Vorovich (1963) and Kosmodamianskii and Shaldurvan (1978).

A special feature of the present investigation lies in the fact that the one-dimensional singular integral equations, or more precisely, an infinite system of such equations, are invoked for the solution of the three-dimensional boundary value problem for a cylinder. Our numerical investigations showed that a solution of the above system by the reduction method converges rapidly over the range of the variation in the thickness coordinate in the skew-symmetric case (bending of a layer). This convergence occurs everywhere in the layer except for areas that are adjacent to its bases in the symmetric case (stretching of the layer). Thus, the proposed procedure decreases the problem dimensionality by a factor of two. In the neighborhood of the base (for the symmetric case) the obtained solution needs some sharpening associated with taking the stiffener singularity into account. This issue is not discussed in this paper, but the experimental investigation of the above phenomenon was given by Villareal et al. (1975).

2. A stretching of a layer with a tunnel crack

In the coordinate system $Ox_1x_2\bar{x}_3$, consider an elastic layer $|\bar{x}_3| \leqslant h$, $\bar{x}_3 = hx_3$ and $-\infty < x_1,\ x_2 < \infty$, weakened by the tunnel through-thickness cuts-cavities (see Fig. 1). The cross sections of the latter represent some smooth open arcs L_j $(j = 1, 2, \ldots, N)$. Let the layer bases be free of forces and allow some loading X_n^+ $(X_n^+ = -X_n^- = X_n, n = 1, 2, 3)$ to be applied on the cavity surfaces. Let a homogeneous field of a stretching and shearing $\langle\sigma_{ij}\rangle$ take place on infinity. It will be also assumed that the curvatures of the arcs

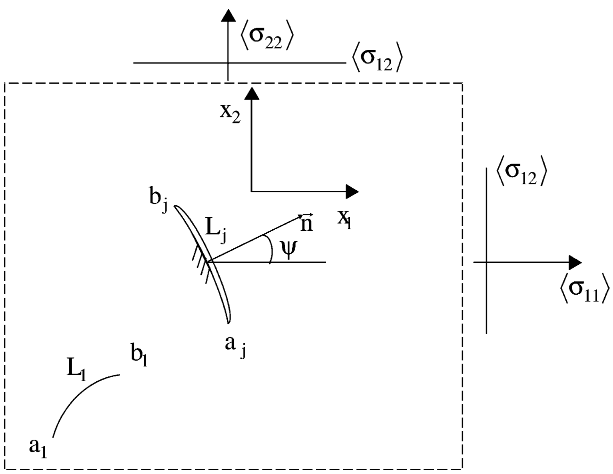

Fig. 1. The layer with tunnel stress raisers.

and functions X_n satisfy Hölder's condition on L_j for any $\bar{x}_3 \in [-h, h]$. In addition, the functions $X_n = X_n(\zeta, \bar{x}_3)$, $\zeta \in L = \bigcup L_j$ are expended into Fourier's series over the coordinate x_3 on $[-1, 1]$.

We will proceed from the homogeneous solutions introduced by Lur'ye (1942). They can be represented in the following complex form:

Biharmonic solution

$$
\begin{aligned}
&u_1 - \mathrm{i}u_2 = -2h\left(\frac{\partial F}{\partial z} + \frac{\partial F_1}{\partial \bar{z}}\right), \quad u_3 = -h^2(\sigma - 1)x_3\nabla^2\varphi,\\
&\sigma_1 = 2\mu h\nabla^2 F, \quad \sigma_2 = 8\mu h\frac{\partial^2 F}{\partial z^2}, \quad \sigma_3 = 0,\\
&F = (3\sigma - 1)\varphi + \frac{h^2}{2}(\sigma - 1)\left(\frac{1}{3} - x_3^2\right)\nabla^2\varphi, \quad \nabla^2\nabla^2\varphi = 0,\\
&\partial_2^2 F_1 = -\partial_1^2 F_1 = 2\sigma\nabla^2\varphi.
\end{aligned}
\tag{2.1}
$$

Rotational solution

$$
\begin{aligned}
&u_1 - \mathrm{i}u_2 = 4\mathrm{i}\sigma h\sum_{m=1}^{\infty}\frac{\partial\varphi_m}{\partial z}\cos\rho_m x_3, \quad u_3 = 0,\\
&\sigma_1 = 0, \quad \sigma_2 = -16\mathrm{i}\mu\sigma h\sum_{m=1}^{\infty}\frac{\partial^2\varphi_m}{\partial z^2}\cos\rho_m x_3, \quad \sigma_{33} = 0,\\
&\sigma_3 = -4\mathrm{i}\mu\sigma\sum_{m=1}^{\infty}\rho_m\frac{\partial\varphi_m}{\partial z}\sin\rho_m x_3,\\
&\rho_m = \pi m, \quad \gamma_m = \rho_m/h, \quad (\nabla^2 - \gamma_m^2)\varphi_m = 0.
\end{aligned}
\tag{2.2}
$$

Potential solution

$$
\begin{aligned}
&u_1 - \mathrm{i}u_2 = 2h\frac{\partial}{\partial z}\mathrm{Re}\sum_{k=1}^{\infty}\alpha_k(x_3)\psi_k, \quad u_3 = \mathrm{Re}\sum_{k=1}^{\infty}\mu_k(x_3)\psi_k,\\
&\sigma_1 = \frac{2\mu}{h}\mathrm{Re}\sum_{k=1}^{\infty}\beta_k(x_3)\psi_k, \quad \sigma_2 = -8\mu h\frac{\partial^2}{\partial z^2}\mathrm{Re}\sum_{k=1}^{\infty}\alpha_k(x_3)\psi_k,\\
&\sigma_3 = 4\mu\sigma\frac{\partial}{\partial z}\mathrm{Re}\sum_{k=1}^{\infty}\kappa_k(x_3)\psi_k, \quad \sigma_{33} = \frac{2\mu\sigma}{h}\mathrm{Re}\sum_{k=1}^{\infty}v_k(x_3)\psi_k,\\
&\alpha_k(x_3) = \left(\frac{t_k}{\delta_k} - \sigma\tau_k\right)\cos\delta_k x_3 - \sigma x_3 t_k\sin\delta_k x_3,\\
&\beta_k(x_3) = \left((2\sigma - 1)\delta_k t_k - \sigma\delta_k^2\tau_k\right)\cos\delta_k x_3 - \sigma\delta_k^2 x_3 t_k\sin\delta_k x_3,\\
&\kappa_k(x_3) = \delta_k(\tau_k\sin\delta_k x_3 - x_3 t_k\cos\delta_k x_3), \quad \mu_k(x_3) = 2\sigma\kappa_k(x_3) - \alpha_k(x_3),\\
&v_k(x_3) = \left(\delta_k t_k + \delta_k^2\tau_k\right)\cos\delta_k x_3 + x_3\delta_k^2 t_k\sin\delta_k x_3,\\
&t_k = \sin\delta_k, \quad \tau_k = \cos\delta_k, \quad \mathrm{Re}\,\delta_k > 0, \quad \mathrm{Im}\,\delta_k > 0,\\
&\lambda_k = \delta_k/h, \quad (\nabla^2 - \lambda_k^2)\psi_k = 0,
\end{aligned}
\tag{2.3}
$$

$$
\begin{aligned}
&z = x_1 + \mathrm{i}x_2, \quad \bar{x}_3 = hx_3, \quad \partial/\partial z = \frac{1}{2}(\partial_1 - \mathrm{i}\partial_2), \quad \partial/\partial\bar{z} = \frac{1}{2}(\partial_1 + \mathrm{i}\partial_2), \quad \partial_i = \partial/\partial x_i,\\
&\sigma = (1 - 2v)^{-1}, \quad \sigma_1 = \sigma_{11} + \sigma_{22}, \quad \sigma_2 = \sigma_{22} - \sigma_{11} + 2\mathrm{i}\sigma_{12}, \quad \sigma_3 = \sigma_{13} - \mathrm{i}\sigma_{23}.
\end{aligned}
$$

In the relationships (2.1)–(2.3), the values u_m and σ_m are components of the displacement vector and stress tensor, respectively; μ and v are the shear modulus and Poisson's ratio, respectively; δ_k are roots of the equation $\sin 2\delta_k + 2\delta_k = 0$ lying in the first quadrant of the complex δ-plane.

The integral representations of functions entering into the relationships (2.1)–(2.3) must provide the existence of jumps in the displacements and a continuity of the stress vector in the transition through L_j $(j = 1, 2, \ldots, N)$ as well as a decay in the displacements and stresses on an infinity. So, it is necessary to add the value $\langle\sigma_{ij}\rangle$ to the components of the stress tensor σ_{ij}. Let us construct below such representations that will be correct in the above-mentioned sense.

Assume

$$\begin{aligned}
&\varphi(z) = \mathrm{Re}\int_L \left(p(\zeta)\frac{\partial G}{\partial\zeta} + p^*(\zeta)\frac{\partial}{\partial\zeta}\nabla^2 G\right)\mathrm{d}\zeta + \int_L q(\zeta)\nabla^2 G\,\mathrm{d}s,\\
&F_1(z) = -4\sigma\mathrm{Re}\int_L p(\zeta)\left[\ln\frac{\zeta - z}{h} - 1\right](\zeta - z)\,\mathrm{d}\zeta,\\
&\psi_k(z) = \int_L q_k(\zeta)K_0(\lambda_k r)\,\mathrm{d}s + \int_L \left(p_k(\zeta)\frac{\partial}{\partial\zeta}K_0(\lambda_k r)\,\mathrm{d}\zeta + p_k^*(\zeta)\frac{\partial}{\partial\zeta}K_0(\lambda_k r)\,\mathrm{d}\bar{\zeta}\right),\\
&\varphi_m(z) = \int_L q_m^*(\zeta)K_0(\gamma_m r)\,\mathrm{d}s + 2\mathrm{Re}\int_L r_m(\zeta)K_0(\gamma_m r)\,\mathrm{d}\zeta,\\
&G = r^2\ln\frac{r}{h}, \quad r = |\zeta - z|, \quad \zeta = \zeta_1 + \mathrm{i}\zeta_2 \in L, \quad \mathrm{Im}\,q_m^*(\zeta) = 0.
\end{aligned} \tag{2.4}$$

Here $K_0(z)$ is the McDonald function of zero order; the functions $p(\zeta) = \{p_j(\zeta), \zeta \in L_j\}, \ldots, r_m(\zeta) = \{r_{mj}(\zeta), \zeta \in L_j\}$ are to be determined from the boundary conditions on the surfaces of non-homogeneities.

The next step of the solution is the reduction of the given boundary value problem to a system of one-dimensional singular integral equations. To complete this step it is necessary to bring a denumerable set of unknown functions appearing in Eqs. (2.1)–(2.3) to one-to-one correspondence with the denumerable set of the integral equations. Some version of the above problem was considered in the variational formulation by Aksentyan and Vorovich (1963). In this paper, we will formulate the problem by focusing other considerations. As a result of the analytical procedure introduced below, all the unknown functions are expressed through some physical values of jumps in the displacement vector on a cut.

From the realization of the above-mentioned procedure, we can expand all the even components of the displacement vector, stress tensor, and external loading into Fourier's series of the type $u = \sum u^{(m)}\cos\rho_m x_3$ and all the odd components into the series of the type $v = \sum v^{(m)}\sin\rho_m x_3$. In determining Fourier's coefficients $\sigma_{ij}^{(m)}$ of the stress tensor components through the functions (2.4), some singularities of the type $(\zeta - z)^{-3}$ will appear. In order to remove the above singularities, we will introduce the following relations between the densities appearing in the representations (2.4):

$$\begin{aligned}
&4(1 - 3\sigma)p^*(\zeta) = \sum_{k=1}^{\infty}\left(\alpha_k^{(0)}p_k(\zeta) + \overline{\alpha_k^{(0)}}\,\overline{p_k^*(\zeta)}\right),\\
&\frac{8(-1)^m(\sigma - 1)h^2}{\pi^2 m^2}p(\zeta) - 4\mathrm{i}\sigma r_m(\zeta) = \sum_{k=1}^{\infty}\left(\alpha_k^{(m)}p_k(\zeta) + \overline{\alpha_k^{(m)}}\,\overline{p_k^*(\zeta)}\right),
\end{aligned} \tag{2.5}$$

where $\alpha_k^{(m)}$ are Fourier's coefficients of the function $\alpha_k(x_3)$ in the orthogonal system $\{\cos\rho_m x_3\}$ on the interval $[-1, 1]$.

Represent the boundary conditions on L in the following form:

$$\begin{aligned} &\left\{\sigma_1^{(m)} - e^{2i\psi}\sigma_2^{(m)}\right\}^{\pm} = \pm 2e^{i\psi}\left\{X_1^{(m)} - iX_2^{(m)}\right\}^{\pm} \quad (m = 0, 1, \ldots), \\ &\mathrm{Re}\left\{e^{i\psi}\left(\sigma_3^{(m)}\right)^{\pm}\right\} = \pm\left(X_3^{(m)}\right)^{\pm} \quad (m = 1, 2, \ldots), \end{aligned} \tag{2.6}$$

where $\sigma_k^{(m)}$ and $X_n^{(m)}$ are the Fourier's coefficients of the functions σ_k and X_n, respectively, in the system $\{\cos\rho_m x_3\}$; the upper sign refers to the left shore of the cut in moving from the vertex a_j to b_j; ψ is the angle between the normal to the left shore and the x_1-axis (Fig. 1).

The continuity conditions for the stress vector in the transition through the cut (it is sufficient in this case to satisfy the boundary equalities (2.6) on only one of the cut shores) and requirements of an existence of a jump in the displacement vector on L will yield to the following relationships:

$$\begin{aligned} &q_m^*(\zeta) = -\frac{\left[u_s^{(m)}\right]}{4\pi\sigma h}, \quad q(\zeta) = \frac{\mathrm{Re}\sum_{k=1}^{\infty}\alpha_k^{(0)}q_k(\zeta)}{4(1-3\sigma)}, \quad p(\zeta) = -\frac{U^{(0)} + iV^{(0)}}{8\pi\sigma h}, \\ &U^{(m)} = \frac{d\left[u_n^{(m)}\right]}{ds} - \frac{\left[u_s^{(m)}\right]}{\rho}, \quad V^{(m)} = \frac{d\left[u_s^{(m)}\right]}{ds} + \frac{\left[u_n^{(m)}\right]}{\rho}, \quad r_m = \frac{ih\left(U^{(m)} + iV^{(m)}\right)}{2\sigma\pi^3 m^2}, \end{aligned} \tag{2.7}$$

$$\begin{cases} \mathrm{Re}\sum\limits_{k=1}^{\infty}\alpha_k^{(m)}q_k = \dfrac{\left[u_n^{(m)}\right]}{2\pi h} \quad (m = 1, 2, \ldots), \\ \mathrm{Re}\sum\limits_{k=1}^{\infty}\kappa_k^{(m)}q_k = 0, \end{cases} \tag{2.8}$$

$$\begin{cases} \mathrm{Re}\sum\limits_{k=1}^{\infty}\alpha_k^{(m)}\left(p_k + p_k^*\right) = \dfrac{2h}{\pi^3 m^2}\left(U^{(m)} - \dfrac{\sigma - 1}{2\sigma}(-1)^{(m)}U^{(0)}\right), \\ \mathrm{Re}\sum\limits_{k=1}^{\infty}\kappa_k^{(m)}\left(p_k + p_k^*\right) = -\dfrac{h}{\pi^2 m\sigma}U^{(m)} \quad (m = 1, 2, \ldots), \end{cases} \tag{2.9}$$

$$\begin{cases} \mathrm{Im}\sum\limits_{k=1}^{\infty}\Lambda_k^{(m)}\left(p_k - p_k^*\right) = -\dfrac{h}{\pi^3 m^2\sigma}V^{(m)}, \quad \Lambda_k^{(m)} = \dfrac{\kappa_k^{(m)}}{\pi m}, \\ \mathrm{Im}\sum\limits_{k=1}^{\infty}\mu_k^{(m)}\left(p_k - p_k^*\right) = \dfrac{\left[u_3^{(m)}\right]}{\pi} - \dfrac{h(\sigma - 1)}{\pi^2 m\sigma}(-1)^{(m)}V^{(0)} \quad (m = 1, 2, \ldots). \end{cases} \tag{2.10}$$

Here $u_3^{(m)}$, $u_n^{(m)}$, and $u_s^{(m)}$ are the Fourier's coefficients of the displacement u_3 and the normal and tangential displacement components on L (in the x_1Ox_2-plane), respectively.

Thus, the densities q_m^*, $p(\zeta)$, and $r_m(\zeta)$ are directly expressed via jumps in the displacements on the cuts. The remaining densities are related to the corresponding jumps by means of three couples of infinite systems of the linear algebraic equations (2.8)–(2.10).

Introducing the representations

$$q_k(\zeta) = \frac{1}{2\pi h}\sum_{j=1}^{\infty} q_{kj}\left[u_n^{(j)}\right],$$

$$p_k + p_k^* = \frac{2h}{\pi^2}\sum_{j=1}^{\infty}\left\{q_{kj}\frac{1}{\pi j^2}\left(U^{(j)} - (-1)^j\frac{\sigma - 1}{2\sigma}U^{(0)}\right) + \frac{q_{kj}^*}{2\sigma j}U^{(j)}\right\},$$

$$p_k - p_k^* = \frac{1}{\pi^2}\sum_{j=1}^{\infty}\left\{\frac{h}{\pi\sigma j^2}S_{kj}V^{(j)} + S_{kj}^*\left(\frac{\left[u_3^{(j)}\right]}{j} - \frac{(-1)^j h(\sigma-1)}{\pi\sigma j^2}V^{(0)}\right)\right\}, \tag{2.11}$$

we will obtain instead of Eqs. (2.8)–(2.10) the following "standard" infinite systems:

$$\begin{cases} \mathrm{Re}\sum_{k=1}^{\infty}\alpha_k^{(m)}q_{kj} = \delta_{mj} \quad (m,j=1,2,\ldots), \\ \mathrm{Re}\sum_{k=1}^{\infty}\kappa_k^{(m)}q_{kj} = 0, \end{cases} \tag{2.12}$$

$$\begin{cases} \mathrm{Re}\sum_{k=1}^{\infty}\alpha_k^{(m)}q_{kj}^* = 0 \quad (m,j=1,2,\ldots), \\ \mathrm{Re}\sum_{k=1}^{\infty}\kappa_k^{(m)}q_{kj}^* = -\delta_{mj}, \end{cases} \tag{2.13}$$

$$\begin{cases} \mathrm{Im}\sum_{k=1}^{\infty}\Lambda_k^{(m)}S_{kj} = -\delta_{mj} \quad (m,j=1,2,\ldots), \\ \mathrm{Im}\sum_{k=1}^{\infty}\mu_k^{(m)}S_{kj} = 0, \end{cases} \tag{2.14}$$

$$\begin{cases} \mathrm{Im}\sum_{k=1}^{\infty}\Lambda_k^{(m)}S_{kj}^* = 0 \quad (m,j=1,2,\ldots), \\ \mathrm{Im}\sum_{k=1}^{\infty}\mu_k^{(m)}S_{kj}^* = \delta_{mj}, \end{cases} \tag{2.15}$$

where the values q_{kj}, q_{kj}^*, S_{kj}, and S_{kj}^* are to be determined; δ_{mj} is the Kronecker's symbol.

Let us go into Eq. (2.12) in a more detail. Multiplying the first system by $\cos\rho_m x_3$ and the second one by $\sin\rho_m x_3$ and summing the results over m, yields:

$$\sum_{k=1}^{\infty}q_{kj}\left(\alpha_k(x_3) - \alpha_k^{(0)}\right) = f_j, \quad \sum_{k=1}^{\infty}q_{kj}\kappa_k(x_3) = 0, \quad f_j = 2\cos\rho_j x_3. \tag{2.16}$$

The pair $\alpha_k(x_3)$ and $\mu_k(x_3)$ is solutions of the following (not self-conjugate) boundary value problem:

$$\begin{aligned} &\alpha_k''(x_3) + (1+\sigma)\delta_k^2\alpha_k(x_3) + \sigma\mu_k'(x_3) = 0, \\ &(1+\sigma)\mu_k''(x_3) + \delta_k^2\mu_k(x_3) + \sigma\delta_k^2\alpha_k'(x_3) = 0, \\ &\alpha_k'(\pm 1) + \mu_k(\pm 1) = 0, \quad (\sigma-1)\delta_k^2\alpha_k(\pm 1) + (\sigma+1)\mu_k'(\pm 1) = 0. \end{aligned} \tag{2.17}$$

Using the above relationships, one can reduce the functional equations (2.16) to the following equivalent form:

$$\sum_{k=1}^{\infty}q_{kj}Y_k(x_3) = -\frac{4\sigma}{\sigma+1}f_j''(x_3), \quad \sum_{k=1}^{\infty}q_{kj}\delta_k^2Y_k(x_3) = \frac{8\sigma}{\sigma+1}f_j^{(4)}(x_3), \tag{2.18}$$

where the functions $Y_k(x_3)$ are non-trivial solutions of the following boundary value problem:

$$Y_k^{(4)} + 2\delta_k^2Y_k'' + \delta_k^4Y_k = 0, \quad Y_k(\pm 1) = Y_k'(\pm 1) = 0. \tag{2.19}$$

Hence, expressions (2.18) represent the expansions of their right-hand sides into Fourier's series for the eigen-functions of problem (2.19). By employing the orthogonality condition (Grinberg, 1953) here, we obtain:

$$\int_{-1}^{1}\left\{2Y_k'Y_m' - \left(\delta_k^2+\delta_m^2\right)Y_kY_m\right\}\mathrm{d}x_3 = 0 \quad (m \neq k),$$

and using the developed equations in the above-mentioned reference procedure, one can obtain Fourier's coefficients q_{kj} $(k, j = 1, 2, \ldots)$, as follows:

$$q_{kj} = \frac{2\sigma\delta_k^2}{\sigma+1}\int_{-1}^{1} f_j''(x_3)Y_k(x_3)\,\mathrm{d}x_3\left[\int_{-1}^{1}\left(Y_k'^2 - \delta_k^2 Y_k\right)\mathrm{d}x_3\right]^{-1}.$$

Calculating the integrals on the right-hand side of the above equation, we will obtain

$$q_{kj} = (-1)^{j+1}\frac{4\pi^2 j^2\delta_k^2}{(\sigma+1)\left(l_{kj}\tau_k\right)^2}, \quad l_{kj} = \delta_k^2 - \rho_j^2, \quad \tau_k = \cos\delta_k. \tag{2.20}$$

Similarly, we can obtain the solutions of the "standard" systems (2.13)–(2.15), as follows:

$$q_{kj}^* = (-1)^{j+1}\frac{2\pi}{\left(l_{kj}\tau_k\right)^2}\left(\pi^2 j^2 - \frac{3\sigma+1}{\sigma+1}\delta_k^2\right), \quad S_{kj} = \mathrm{i}\left(2\sigma q_{kj} + \pi j q_{kj}^*\right), \quad S_{kj}^* = \mathrm{i}q_{kj}. \tag{2.21}$$

Thus, a solvability of the infinite systems of equations (2.12)–(2.15) has been established. The closed-form solutions are obtained by Eqs. (2.20) and (2.21). Moreover, all the densities in integral representations (2.4) are expressed through the physical values, namely, through the jumps in the "displacements" on L.

Eq. (2.11) can be significantly simplified by substituting the coefficients from (2.20) and (2.21) into the above equations and then by summing up some series. As a result, we will obtain:

$$\begin{aligned}
-\frac{\pi^3\sigma}{h}\left(p_k + p_k^*\right) &= \varepsilon_k U^{(0)} + \mathrm{i}\sum_{j=1}^{\infty}\frac{S_{kj}}{j^2}U^{(j)},\\
-\frac{\pi\sigma}{h}\left(p_k - p_k^*\right) &= \mathrm{i}\varepsilon_k V^{(0)} - \sum_{j=1}^{\infty}\frac{S_{kj}}{j^2}V^{(j)} - \frac{\mathrm{i}\pi\sigma}{h}\sum_{j=1}^{\infty}\frac{q_{kj}}{j}\left[u_3^{(j)}\right],\\
\varepsilon_k &= \frac{2(\sigma-1)\pi^2}{(\sigma+1)\delta_k^2\tau_k^2}.
\end{aligned} \tag{2.22}$$

Notice that representations (2.4) also remain valid for the second basic problem also, for instance, in the presence of a rigid insertion in the cut. However, in this case the densities will be expressed via some jumps in the stress vector on L.

2.1. The integral equations of the boundary value problem (2.6)

The integral representations for stresses σ_{ij} can be obtained by substituting for functions in relationships (2.1)–(2.3) from Eq. (2.4) into the above relationships. Expanding the obtained expressions into the Fourier's series in the coordinate x_3, we will determine the integral representations of Fourier's coefficients $\sigma_{ij}^{(m)}$. Satisfying the boundary conditions (2.6) on one of the shores L and taking Eqs. (2.5), (2.7), (2.11) and (2.22) into consideration, we will obtain an infinite system of one-dimensional singular integro-differential equations of the first kind. Because of its awkwardness, the above system is not given here.

The structure of the system is such that all unknowns are involved in its regular part; the characteristic part of the system contains exactly three functions $\lfloor u_n^{(m)} \rfloor$, $\lfloor u_s^{(m)} \rfloor$, $\lfloor u_3^{(m)} \rfloor$ for any fixed $m = 1, 2, \ldots$, and two functions $\lfloor u_n^{(0)} \rfloor$ and $\lfloor u_s^{(0)} \rfloor$ for $m = 0$.

The jumps in displacements vanish at the vertices of arcs L_j, therefore the above-obtained system should be considered together with the following additional conditions:

$$\int_{L_j} \left(U^{(m)} + \mathrm{i}V^{(m)} \right) \mathrm{d}\zeta = 0, \quad \int_{L_j} \mathrm{d}\left\lfloor u_3^{(m)} \right\rfloor = 0, \quad j = 1, 2, \ldots, N, \; m = 0, 1, \ldots \tag{2.23}$$

The functions $U^{(m)}$, $V^{(m)}$, and $\mathrm{d}\lfloor u_3^{(m)} \rfloor / \mathrm{d}s$ are found in the class h_0 (Muskhelishvili, 1958).

Consider a characteristic part of the above-mentioned system in a detail. For simplicity, assume that the contour L represents a segment $x_2 = 0$, $-l \leqslant x_1 \leqslant l$. Then, we obtain:

for $m = 0$

$$\int_{-l}^{l} \frac{\mathrm{d}\left[u_2^{(0)} + \mathrm{i}u_1^{(0)} \right]}{x - x_0} = N_0(x_0), \quad -l < x_0 < l; \tag{2.24}$$

for $m = 1, 2, \ldots$

$$\int_{-l}^{l} \frac{\mathrm{d}\left[u_2^{(m)} \right]}{x - x_0} = N_m(x_0), \quad -l < x_0 < l, \tag{2.25}$$

$$\int_{-l}^{l} y_{jm}(x) \frac{\mathrm{d}x}{x - x_0} = N_{jm}(x_0) \quad (j = 1, 2), \tag{2.26}$$

where

$$y_{jm}(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left[u_1^{(m)} \right] + \alpha_m \left[u_3^{(m)} \right], \quad \alpha_m = \frac{\sigma - 1}{2\sigma} \gamma_m, \quad y_{2m}(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left[u_3^{(m)} \right] - \gamma_m \left[u_1^{(m)} \right].$$

The functions $N_m(x)$, $N_{jm}(x) \in H[-1, 1]$ are assumed to be known. Eqs. (2.24) and (2.25) are solvable, their solutions are fixed by additional conditions of the type (2.23). By replacing

$$\omega_{1m} = \frac{\mathrm{d}}{\mathrm{d}x}\left[u_1^{(m)} \right], \quad \omega_{2m} = \frac{\mathrm{d}}{\mathrm{d}x}\left[u_3^{(m)} \right],$$

the remaining system (2.26) is easily reduced to the standard form (Muskhelishvili, 1958 and Parton and Perlin, 1973):

$$\int_{-l}^{l} \frac{\omega_{1m}\, \mathrm{d}x}{x - x_0} + \alpha_m \int_{-l}^{l} \ln \left| \frac{l - x_0}{x - x_0} \right| \mathrm{d}x = N_{1m}(x_0) \quad (m = 1, 2, \ldots),$$
$$\int_{-l}^{l} \frac{\omega_{2m}\, \mathrm{d}x}{x - x_0} - \gamma_m \int_{-l}^{l} \omega_{1m} \ln \left| \frac{l - x_0}{x - x_0} \right| \mathrm{d}x = N_{2m}(x_0),$$

where the kernels in the second terms on the left-hand sides are regular. Thus, the characteristic part of the obtained system is solvable in the class h_0 for any fixed $m = 0, 1, \ldots$

2.2. The stress intensity factors

Introduce parametrization of the contour L_j (from here on, the index j will be dropped) $\zeta = \zeta(\delta)$, $\zeta_0 = \zeta(\delta_0)$, $-1 \leqslant \delta$, $\delta_0 \leqslant 1$. Accordingly to that we will set:

$$\omega_p^{(m)}(\zeta) = \frac{\Omega_p^{(m)}(\delta)}{s'(\delta)\sqrt{1-\delta^2}} \quad (p = 1, 2;\ m = 0, 1, \ldots), \quad \frac{d\omega_3^{(m)}}{ds} = \frac{\Omega_3^{(m)}(\delta)}{s'(\delta)\sqrt{1-\delta^2}}, \quad s'(\delta) = \frac{ds}{d\delta} > 0,$$

where functions $\Omega_p^{(m)}(\delta) \in H[-1, 1]$ are the solutions of the obtained system of integral equations for the boundary value problem.

Using the above expressions and the relations connecting the densities in the integral representations (2.4) with jumps in the displacements, as well as Eqs. (2.1)–(2.3), we will find after a detailed asymptotic analysis of the integral representations for the stresses the following:

$$K_{\rm I} - {\rm i}K_{\rm II} = -\frac{\mu\sigma}{\sigma+1}\sqrt{\frac{\pi}{s'(\pm 1)}}\sum_{m=0}^{\infty}\left\{\Omega_1^{(m)}(\pm 1) - {\rm i}\Omega_2^{(m)}(\pm 1)\right\}\cos m\pi x_3,$$

$$K_{\rm III} = -\frac{\mu h}{2}\sqrt{\frac{\pi}{s'(\pm 1)}}\sum_{m=1}^{\infty}\Omega_3^{(m)}(\pm 1)\sin m\pi x_3, \quad K_{\rm I} = \sqrt{2\pi r}\sigma_n, \quad K_{\rm II} = \sqrt{2\pi r}\sigma_{ns}, \quad K_{\rm III} = \sqrt{2\pi r}\sigma_{n3},$$

where σ_n, σ_{ns}, and σ_{n3} are the normal and shear stresses on a plane beyond the crack tip (the upper sign corresponds to the origin of the crack "a"). Thus, the stress intensity factors are naturally expressed through the functions:

$$U = \frac{d}{ds}[u_n(\zeta, x_3)] - \frac{1}{\rho}[u_s(\zeta, x_3)], \quad \frac{d}{ds}[u_3(\zeta, x_3)],$$

$$V = \frac{d}{ds}[u_s(\zeta, x_3)] + \frac{1}{\rho}[u_n(\zeta, x_3)], \quad \left(\zeta = a_j \cup b_j, x_3 \in [-1, 1]\right).$$

2.3. Some numerical results

For example, consider a layer weakened by a tunnel parabolic cut $\xi_1 = p_1\delta$, $\xi_2 = p_2\delta^2$ $(-1 \leqslant \delta \leqslant 1)$ subjected to homogeneous stress field $\langle\sigma_{ij}\rangle$ on infinity. Some load X_n $(n = 1, 2, 3)$ can be applied on the cut surface.

In the numerical implementation of the algorithm, the system of integral equations was reduced to the system of linear algebraic equations by the mechanical quadrature method (see Erdogan et al. (1973)) to the N_*th approximation. In the zero approximation, two integral equations are retained, and correspondingly, two densities $\lfloor u_1^{(0)} \rfloor$ and $\lfloor u_2^{(0)} \rfloor$. Analyses were carried out for $N_* = 0, 1, 2, 3, 4$ where the third approximation did not contribute the results for the values $K_{\rm I}$ and $K_{\rm II}$ in the interval $|x_3| < 0.85$. For the value of $K_{\rm III}$ faster convergence in the interval $|x_3| \leqslant 1$ was observed.

Let $\langle\sigma_{22}\rangle \neq 0$, $\langle\sigma_{11}\rangle = \langle\sigma_{12}\rangle = X_n = 0$ $(n = 1, 2, 3)$. The diagrams of the distribution of the relative stress intensity factor $\langle K_1\rangle = K_{\rm I}/(\langle\sigma_{22}\rangle\sqrt{\pi l})$ are given in Fig. 2 along the thickness coordinate for various p_2 and h/l ($2l$ is the crack length). The top three diagrams refer to a straight crack ($p_2 = 0$) and the bottom-to a parabolic crack ($p_2 = 0.5$). For $p_2 = 1$ these three curves are very close to one another and to the mark $\langle K_{\rm I}\rangle = 0.2$. Here and below, in the analysis, $p_1 = 1$ and $\nu = 0.3$ were taken.

Now let $\langle\sigma_{12}\rangle \neq 0$, $\langle\sigma_{11}\rangle = \langle\sigma_{22}\rangle = X_n = 0$. Notice that in this case, the value $\langle K_{\rm II}\rangle$ does not practically depend on the coordinate x_3. The values $\langle K_{\rm II}\rangle$ for a straight crack vary in the limits 1.020–1.013 and 1.030–1.024 for $h/l = 2.5$ and 1, respectively. For a parabolic crack, the values $\langle K_{\rm II}\rangle$ for $h/l = 0.5$; 1; 2, 5 are approximately equal. For $p_2 = 0.5$, $\langle K_{\rm II}\rangle \approx 0.47$–$0.48$; for $p_2 = 1$, $\langle K_{\rm II}\rangle = 0.08$–$0.10$. Consider one more case

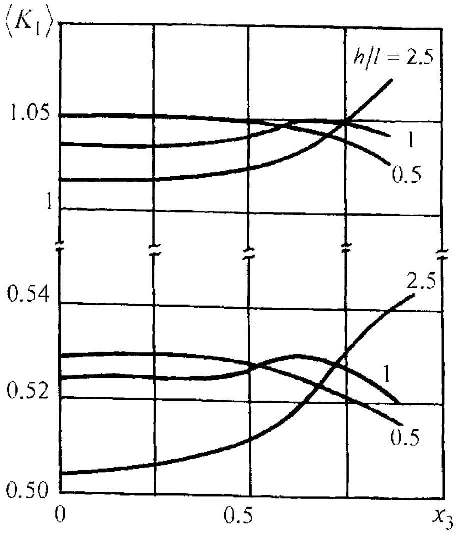

Fig. 2. Distribution of the relative stress intensity factor $\langle K_{\mathrm{I}} \rangle$ along the thickness coordinate x_3 for different h/l.

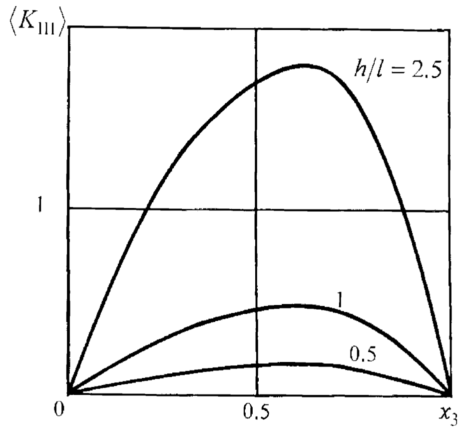

Fig. 3. Distribution of the relative stress intensity factor $\langle K_{\mathrm{III}} \rangle$ along the thickness coordinate x_3 for different h/l.

when $\langle \sigma_{ij} \rangle = 0$ and the shear forces $X_3 = X_3^{(1)} \sin \pi x_3, X_1 = X_2 = 0$ on the crack surface. The diagrams of the value $\langle K_{\mathrm{III}} \rangle = K_{\mathrm{III}} (X_3^{(1)} \sqrt{\pi l})^{-1}$ are shown in Fig. 3.

3. Bending of a layer with a tunnel crack

Now consider a problem of the bending of a layer. It should be noted that a contact of crack surfaces is not taken into account here because it is assumed that the layer was subjected to stretching prior to bending. Thus the contact is lacking in this case. Let the homogeneous field of bending and twisting stresses $\langle \sigma_{ij} \rangle$ takes place on infinity. In order to describe the state of stress and strain of the above layer we will apply the homogeneous solutions developed by Lur'ye (1942). We will present these solutions in the following complex form:

Biharmonic solutions

$$
\begin{aligned}
&u_1 - \mathrm{i}u_2 = \frac{\partial}{\partial z}F_1, \quad u_3 = -(\sigma+1)F - \frac{1}{2}h^2x_3^2(\sigma-1)\nabla^2F + 2\sigma h^2\nabla^2F,\\
&\sigma_1 = 2\mu(3\sigma-1)hx_3\nabla^2F, \quad \sigma_{33} = 0, \quad \sigma_2 = -4\mu\frac{\partial^2F_1}{\partial z^2}, \quad \sigma_3 = 4\mu\sigma h^2(1-x_3^2)\frac{\partial}{\partial z}\nabla^2F,\\
&F_1 = 2(\sigma+1)hx_3F - h^3x_3^3\left(\sigma+\frac{1}{3}\right)\nabla^2F, \quad \nabla^2\nabla^2F = 0.
\end{aligned}
\tag{3.1}
$$

Rotational solution

$$
\begin{aligned}
&u_1 - \mathrm{i}u_2 = 4\mathrm{i}\sigma h^2\sum_{m=0}^{\infty}\frac{\sin\rho_m x_3}{\rho_m}\frac{\partial\varphi_m}{\partial z}, \quad u_3 = 0,\\
&\sigma_1 = 0, \quad \sigma_2 = -16\mathrm{i}\mu\sigma h^2\sum_{m=0}^{\infty}\frac{\sin\rho_m x_3}{\rho_m}\frac{\partial^2\varphi_m}{\partial z^2}, \quad \sigma_{33} = 0,\\
&\sigma_3 = 4\mathrm{i}\mu\sigma h\sum_{m=0}^{\infty}\frac{\partial\varphi_m}{\partial z}\cos\rho_m x_3,\\
&\rho_m = (2m+1)\pi/2, \quad \left(\nabla^2-\gamma_m^2\right)\varphi_m = 0, \quad \gamma_m = \rho_m/h.
\end{aligned}
\tag{3.2}
$$

Potential solution

$$
\begin{aligned}
&u_1 - \mathrm{i}u_2 = 2h\frac{\partial}{\partial z}\mathrm{Re}\sum_{k=1}^{\infty}\alpha_k^*(x_3)\psi_k, \quad u_3 = -\mathrm{Re}\sum_{k=1}^{\infty}\mu_k^*(x_3)\psi_k,\\
&\sigma_1 = \frac{2\mu}{h}\mathrm{Re}\sum_{k=1}^{\infty}\beta_k^*(x_3)\psi_k, \quad \sigma_2 = -8\mu h\frac{\partial^2}{\partial z^2}\mathrm{Re}\sum_{k=1}^{\infty}\alpha_k^*(x_3)\psi_k,\\
&\sigma_3 = 4\mu\sigma\frac{\partial}{\partial z}\mathrm{Re}\sum_{k=1}^{\infty}\kappa_k^*(x_3)\psi_k, \quad \sigma_{33} = \frac{2\mu\sigma}{h}\mathrm{Re}\sum_{k=1}^{\infty}v_k^*(x_3)\psi_k,\\
&\alpha_k^*(x_3) = \left(\sigma t_k + \frac{\tau_k}{\delta_k}\right)\sin\delta_k x_3 + \sigma x_3\tau_k\cos\delta_k x_3,\\
&\mu_k^*(x_3) = [(1+\sigma)\tau_k - \sigma\delta_k t_k]\cos\delta_k x_3 + \sigma\delta_k x_3\tau_k\sin\delta_k x_3,\\
&\beta_k^*(x_3) = 2(\sigma-1)\delta_k\tau_k\sin\delta_k x_3 + \delta_k^2\alpha_k^*,\\
&v_k^*(x_3) = (\delta_k\tau_k - \delta_k^2 t_k)\sin\delta_k x_3 - x_3\delta_k^2\tau_k\cos\delta_k x_3,\\
&\kappa_k^*(x_3) = \delta_k(t_k\cos\delta_k x_3 - x_3\tau_k\sin\delta_k x_3),\\
&t_k = \sin\delta_k, \quad \tau_k = \cos\delta_k, \quad (\nabla^2-\lambda_k^2)\psi_k = 0, \quad \lambda_k = \delta_k/h.
\end{aligned}
\tag{3.3}
$$

Here, δ_k are roots of the equation $\sin 2\delta_k - 2\delta_k = 0$ lying in the first quadrant of the complex δ-plane.

The integral representations of the functions in the relationships (3.1)–(3.3) must ensure the existence of displacement jumps and continuity of the stress vector in the transition through the cuts L_j $(j = 1, 2, \ldots, N)$ as well as, the decay of displacements and stresses on infinity. The integral representations may be written as following:

$$
F(z) = \mathrm{Re}\int_L\left(p(\zeta)\frac{\partial G}{\partial\zeta} + p^*(\zeta)\frac{\partial}{\partial\zeta}\nabla^2G\right)\mathrm{d}\zeta + \int_L q(\zeta)\nabla^2G\,\mathrm{d}s + \int_L q^*(\zeta)G\,\mathrm{d}s,
$$

$$\varphi_m(z) = \int_L q_m^*(\zeta) K_0(\gamma_m r)\,\mathrm{d}s + 2\mathrm{Re}\int_L r_m(\zeta)\frac{\partial}{\partial\zeta}K_0(\gamma_m r)\,\mathrm{d}\zeta, \tag{3.4}$$

$$\psi_k(z) = \int_L \left(p_k(\zeta)\frac{\partial}{\partial\zeta}K_0(\lambda_k r)\,\mathrm{d}\zeta + p_k^*(\zeta)\frac{\partial}{\partial\bar\zeta}K_0(\lambda_k r)\,\mathrm{d}\bar\zeta\right) + \int_L q_k^* K_0(\lambda_k r)\,\mathrm{d}s,$$

$$G = r^2 \ln r, \quad L = \bigcup_{j=1}^{N} L_j, \quad \zeta - z = re^{i\alpha}, \quad \mathrm{Im}\, q_m^* = 0, \quad \mathrm{Im}\, q(\zeta) = \mathrm{Im}\, q^*(\zeta) = 0.$$

Here the functions $p(\zeta) = \{p_j(\zeta), \zeta \in L_j\}, \ldots, r_m(\zeta) = \{r_{mj}(\zeta), \zeta \in L_j\}$ are to be determined from the boundary conditions on the surface of the non-homogeneity.

As before, we expand all odd components of the displacement vector and stress tensor, as well as external loading in Fourier's series of the type $u = \sum u^{(m)} \sin \rho_m x_3$ and all the even components- in the series of the type $v = \sum v^{(m)} \cos \rho_m x_3$. In determining the Fourier's coefficients $\sigma_{ij}^{(m)}$ of the components of the stress tensor σ_{ij} by functions (3.4), the singularities of the type $(\zeta - z)^{-3}$ occur in the kernels of the corresponding integral representations. In order to eliminate them, we introduce some relations between the densities appearing in the representations (3.4).

$$\Delta_m^* p^* - \varepsilon_m^* p - \frac{\mathrm{i}\rho_m^*}{2} r_m = h \sum_{k=1}^{\infty}\left(\alpha_k^{(m)} p_k(\zeta) + \bar\alpha_k^{(m)}\overline{p_k^*(\zeta)}\right), \tag{3.5}$$

where

$$\Delta_m^* = 2(\sigma + 1)h\Delta_m, \quad \Delta_m = \frac{2(-1)^m}{\rho_m^2}, \quad \varepsilon_m^* = h^3\left(\sigma + \frac{1}{3}\right)\varepsilon_m,$$

$$\varepsilon_m = \frac{6(-1)^m}{\rho_m^2}\left(1 - \frac{2}{\rho_m^2}\right), \quad \rho_m^* = \frac{4\sigma h^2}{\rho_m},$$

$\alpha_k^{(m)}$ are Fourier's coefficients of the function $\alpha_k^*(x_3)$ in the orthogonal system $\{\sin \gamma_m x_3\}$.

The boundary conditions on L have a form (2.6) for $m = 0, 1, 2, \ldots$ The continuity conditions of the stress vector in the transition through the cut and the requirement of the existence of a jump in the displacement vector on L result in the following relationships:

$$\begin{aligned} q_m^* &= \frac{\rho_m}{4\pi\sigma h^2}\left(\left[u_1^{(m)}\right]\sin\psi - \left[u_2^{(m)}\right]\cos\psi\right), \\ r_m &= \frac{1}{4\pi\sigma h}\left[u_3^{(m)}\right] + \frac{\mathrm{i}}{2\pi\sigma\rho_m}V^{(m)}, \quad V^{(m)} = e^{-\mathrm{i}\psi}\frac{\mathrm{d}}{\mathrm{d}s}\left(\left[u_1^{(m)}\right] + \mathrm{i}\left[u_2^{(m)}\right]\right); \end{aligned} \tag{3.6}$$

$$\begin{cases} \Delta_m^* \mathrm{Re}\, p^* - \varepsilon_m^* \mathrm{Re}\, p - h\mathrm{Re}\sum\limits_{k=1}^{\infty}\alpha_k^{(m)}(p_k + p_k^*) = -\dfrac{h^2}{\pi\rho_m^2}\left(\dfrac{\mathrm{d}}{\mathrm{d}s}\left[u_1^{(m)}\right]\cos\psi + \dfrac{\mathrm{d}}{\mathrm{d}s}\left[u_2^{(m)}\right]\sin\psi\right), \\ \lambda_m^{**}\mathrm{Re}\, p - \sigma\mathrm{Re}\sum\limits_{k=1}^{\infty}\kappa_k^{(m)}(p_k + p_k^*) = -\dfrac{h}{2\pi\rho_m}\left(\dfrac{\mathrm{d}}{\mathrm{d}s}\left[u_1^{(m)}\right]\cos\psi + \dfrac{\mathrm{d}}{\mathrm{d}s}\left[u_2^{(m)}\right]\sin\psi\right); \end{cases} \tag{3.7}$$

$$\begin{cases} -\Delta_m^* \mathrm{Re}\, q + \varepsilon_m^* q^* + h\mathrm{Re}\sum\limits_{k=1}^{\infty}\alpha_k^{(m)} q_k^* = \dfrac{1}{4\pi}\left(\left[u_1^{(m)}\right]\cos\psi + \left[u_2^{(m)}\right]\sin\psi\right), \\ -\lambda_m^{**} q^* + \sigma\mathrm{Re}\sum\limits_{k=1}^{\infty}\kappa_k^{(m)} q_k^* = 0; \end{cases} \tag{3.8}$$

$$\begin{cases} \Delta_m^* \operatorname{Im} p^* - \varepsilon_m^* \operatorname{Im} p - h \operatorname{Im} \sum_{k=1}^{\infty} \alpha_k^{(m)} (p_k - p_k^*) = \frac{h}{2\pi\rho_m} \left[u_3^{(m)} \right] + \frac{h^2}{\pi\rho_m^2} \left(\frac{\mathrm{d}}{\mathrm{d}s} \left[u_1^{(m)} \right] \sin\psi - \frac{\mathrm{d}}{\mathrm{d}s} \left[u_2^{(m)} \right] \cos\psi \right), \\ \Delta_m \operatorname{Im} p - \frac{2\sigma\rho_m}{h} \operatorname{Im} \sum_{k=1}^{\infty} \kappa_k^{(m)} (p_k - p_k^*) = \frac{1}{\pi} \left(\frac{\mathrm{d}}{\mathrm{d}s} \left[u_1^{(m)} \right] \sin\psi - \frac{\mathrm{d}}{\mathrm{d}s} \left[u_2^{(m)} \right] \cos\psi \right); \end{cases} \tag{3.9}$$

$$\lambda_m^{**} = 2\sigma h^2 (\lambda_m^* - \Lambda_m), \quad \lambda_m^* = \frac{2(-1)^m}{\rho_m}, \quad \Lambda_m = 2(-1)^m \left(\frac{1}{\rho_m} - \frac{2}{\rho_m^3} \right),$$
$$\Lambda_m^* = 2\delta_m^{**} + \Delta_m^*, \quad \delta_m^{**} = (3\sigma - 1) h \Delta_m.$$

Here $u_i^{(m)}$ ($i = \overline{1,3}$) are the Fourier's coefficients in the displacement vector u_i ($i = \overline{1,3}$).

Thus, the densities q_m^* and r_m are directly expressed via the jumps in the displacements on the cuts. The remaining densities are connected with the jumps by means of the three pairs of the infinite systems of linear algebraic equations (3.7)–(3.9). Isolating the terms corresponding to the value $m = 0$ in relationships (3.7)–(3.9), one can find the unknown densities, as follows:

$$\begin{aligned} & q^* = -l U^{(0)} + \sum_{j=1}^{\infty} l_j U^{(j)}, \quad q = -l^* U^{(0)} + \sum_{j=1}^{\infty} l_j^* U^{(j)}, \\ & p = \mathrm{i} l_1 \left[u_3^{(0)} \right] - l_2 V^{(0)} - \mathrm{i} \sum_{j=1}^{\infty} l_{11}^{(j)} \left[u_3^{(j)} \right] + \sum_{j=1}^{\infty} l_{12}^{(j)} V^{(j)}, \\ & p^* = \mathrm{i} l_1^* \left[u_3^{(0)} \right] - l_2^* V^{(0)} - \mathrm{i} \sum_{j=1}^{\infty} l_{21}^{(j)} \left[u_3^{(j)} \right] + \sum_{j=1}^{\infty} l_{22}^{(j)} V^{(j)}, \\ & p_k = l_{1k} \left[u_3^{(0)} \right] - l_{2k} V^{(0)} - \sum_{j=1}^{\infty} l_{1k}^{(j)} \left[u_3^{(j)} \right] + \sum_{j=1}^{\infty} l_{2k}^{(j)} V^{(j)}, \\ & p_k^* = -l_{1k} \left[u_3^{(0)} \right] - l_{2k} \overline{V}^{(0)} + \sum_{j=1}^{\infty} l_{1k}^{(j)} \left[u_3^{(j)} \right] + \sum_{j=1}^{\infty} l_{2k}^{(j)} \overline{V}^{(j)}, \\ & p_k^* = -l_{1k} \left[u_3^{(0)} \right] - l_{2k} \overline{V}^{(0)} + \sum_{j=1}^{\infty} l_{1k}^{(j)} \left[u_3^{(j)} \right] + \sum_{j=1}^{\infty} l_{2k}^{(j)} \overline{V}^{(j)}, \\ & q_k^* = -l_k^* U^{(0)} + \sum_{j=1}^{\infty} l_k^{*(j)} U^{(j)}, \end{aligned} \tag{3.10}$$

where $U^{(m)} = [u_1^{(m)}] \cos\psi + [u_2^{(m)}] \sin\psi$, $[u] = u^+(\zeta) - u^-(\zeta)$, $\zeta \in L_j$.

The expressions for the coefficients for the displacement jumps are not given here because of their awkwardness. These coefficients are expressed through the values q_{kj} and q_{kj}^*, which are the solutions of the following "standard" systems:

$$\begin{cases} \operatorname{Re} \sum_{k=1}^{\infty} \alpha_k^{(i)} q_{kj}^* = \delta_{ij}, \\ \operatorname{Re} \sum_{k=1}^{\infty} \kappa_k^{(i)} q_{kj}^* = 0; \end{cases} \qquad \begin{cases} \operatorname{Re} \sum_{k=1}^{\infty} \alpha_k^{(i)} q_{kj} = 0, \\ \operatorname{Re} \sum_{k=1}^{\infty} \kappa_k^{(i)} q_{kj} = \delta_{ij}, \end{cases} \tag{3.11}$$

where δ_{ij} $(i,j=1,2)$ is the Kronecker's symbol. Thus, unknown densities are directly expressed via the displacement jumps. The solvability of the infinite system of equations of the type (3.10) was proven in Section 2.

3.1. The integral equations of the boundary volume problem

The integral representations for the stresses σ_{ij} can be obtained by substituting for the functions from (3.4) into relationships (3.4). Expanding the obtained expressions into Fourier's series in the x_3-coordinate, one can determine the integral representations of the Fourier's coefficients $\sigma_{ij}^{(m)}$. Then, satisfying boundary conditions (2.6) on one of the shores L and taking Eqs. (3.5) and (3.6) into account yields an infinite system of one-dimensional singular integro-differential equations of the first kind. The above system is not given here because of its awkwardness. The system has been constructed in such a way that all the unknowns "are linked" in its regular part; the characteristic part of the system involves exactly three functions $\lfloor u_1^{(m)}\rfloor$, $\lfloor u_2^{(m)}\rfloor$, and $\lfloor u_3^{(m)}\rfloor$ for any fixed value $m=0,1,2,\ldots$

The displacement jumps at the vertices of arcs L_j vanish; therefore it is necessary to consider the obtained system of equations together with the following additional conditions:

$$\int_{L_j} \mathrm{d}\left[u_i^{(m)}\right] = 0 \quad (j=1,2,\ldots,N,\ i=\overline{1,3},\ m=0,1,\ldots), \tag{3.12}$$

and to seek the functions $\mathrm{d}\lfloor u_i^{(m)}\rfloor/\mathrm{d}s$ in the class h_0 (Muskhelishvili, 1958). The solvability of the characteristic part of the obtained system is established similarly to that of Section 2.

3.2. The coefficients of stress intensity

Let us introduce the parameterization of the contour L_j (below the index j will again be omitted) $\zeta=\zeta(\delta)$, $\zeta_0=\zeta_0(\delta)$, $-1\leqslant\delta$, $\delta_0\leqslant 1$ similarly to that of Section 2. Corresponding to that, we will set:

$$\omega_p^{(m)}(\zeta) = \frac{\Omega_p^{(m)}(\delta)}{s'(\delta)\sqrt{1-\delta^2}} \quad (p=1,2;\ m=0,1,\ldots),$$

$$\frac{\mathrm{d}\omega_3^{(m)}}{\mathrm{d}s} = \frac{\Omega_3^{(m)}(\delta)}{s'(\delta)\sqrt{1-\delta^2}}, \quad s'(\delta)=\frac{\mathrm{d}s}{\mathrm{d}\delta}>0,$$

where the function $\Omega_p^{(m)}(\delta)\in H[-1,1]$.

Using the above expressions, the relationships (3.6) and (3.9) connecting the densities in the integral representations (3.4) with the displacement jumps and Eqs. (3.1)–(3.3), we will find by the asymptotic analysis the following integral representations for the stresses:

$$K_{\mathrm{I}} - \mathrm{i}K_{\mathrm{II}} = -\frac{\mu\sigma}{\sigma+1}\sqrt{\frac{\pi}{s'(\mp 1)}}\sum_{m=0}^{\infty}\left\{\Omega_1^{(m)}(\mp 1) - \mathrm{i}\Omega_2^{(m)}(\mp 1)\right\}\sin\rho_m x_3,$$

$$K_{\mathrm{III}} = -\frac{\mu}{2}\sqrt{\frac{\pi}{s'(\mp 1)}}\sum_{m=0}^{\infty}\Omega_3^{(m)}(\mp 1)\cos\rho_m x_3, \quad \rho_m=\frac{2m+1}{2h}\pi,$$

$$K_{\mathrm{I}}=\sqrt{2\pi r}\sigma_n, \quad K_{\mathrm{II}}=\sqrt{2\pi r}\sigma_{ns}, \quad K_{\mathrm{III}}=\sqrt{2\pi r}\sigma_{n3},$$

where σ_n, σ_{ns} and σ_{n3} are the normal and shear stresses on a plane beyond the crack tip; the upper sign refers to the beginning of the crack a.

3.3. Some numerical results

As an example, let us consider a layer weakened by a tunnel parabolic cut $\xi_1 = p_1\delta$, $\xi_2 = p_2\delta^2$ $(-1 \leqslant \delta \leqslant 1)$ is subjected to a homogeneous field of bending stresses $\langle\sigma_{ij}\rangle$ applied on infinity. It is assumed that some bending load $X_n(\zeta, x_3)$ $(n = 1, 2, 3)$ can be applied on the cut surface.

In the numerical implementation of the algorithm, the system of integral equations was reduced to a system of linear algebraic equations by the method of mechanical quadratures (Erdogan et al., 1973) and was then solved by the reduction method. Approximation with the number N_* refers to retaining $3(N_* + 1)$ real equations in the system and correspondingly $3(N_* + 1)$ unknowns $\lfloor u_1^{(m)} \rfloor$, $\lfloor u_2^{(m)} \rfloor$, and $\lfloor u_3^{(m)} \rfloor$ $(m = 0, 1, 2, \ldots, N_*)$. At a zero approximation, the three integral equations were retained and, correspondingly, so were the three densities $\lfloor u_1^{(0)} \rfloor$, $\lfloor u_2^{(0)} \rfloor$, and $\lfloor u_3^{(0)} \rfloor$. The calculations were carried out for $N_* = 0$, 1, 2, 3, 4, where the third approximation did not correct the results in the interval $|x_3| \leqslant 1$.

Let the load $\langle\sigma_{22}\rangle = Px_3$, $P = \text{const}$ acts on an infinity. The surfaces of the cavity-cut are free of any forces. The diagrams of distribution of the relative stress coefficient factor $\langle K_1\rangle = K_1/(P\sqrt{\pi l})$ over the "thickness" coordinate are shown in Fig. 4. The curves 1, 2, 3, and 4 were constructed for a straight crack $(p_1 = 1,\ p_2 = 0.5)$ for $h/l = 0.5$; 1; 2; and 4 respectively, where $2l$ is the crack length. Points on the above figure refer to results given in Sundara Raja Iyengar et al. (1988) and were obtained by the FEM. It should be noted that the results of the given investigation agree well with those of Sundara Raja Iyengar et al. (1988). The straight line in Fig. 4 correspond to results of Murthy et al. (1981) obtained with the use of the Reissner's theory for $h/l = 1$.

Let the load $X_1 = Px_3 \cos\psi$, $X_2 = Px_3 \sin\psi$, $X_3 = 0$, $P = \text{const}$, and $\langle\sigma_{ij}\rangle = 0$ be applied on the surface of the cavity-cut. The corresponding diagrams of the distribution of the relative stress intensity factor $\langle K_1\rangle$ over the "thickness" coordinate are given in Fig. 5. The curves 1, 2, 3, and 4 refer to a parabolic crack $(p_1 = 1$ and $p_2 = 0.5)$ for $h/l = 0.5$; 1; 2; and 4, respectively.

Consider a case when a load $X_1 = X_2 = 0$, $X_3 = P$ acts on the surface of the cavity of the cut. The diagrams of the distribution of the relative stress concentration factor $\langle K_{\text{III}}\rangle = K_{\text{III}}/(P\sqrt{\pi l})$ over the

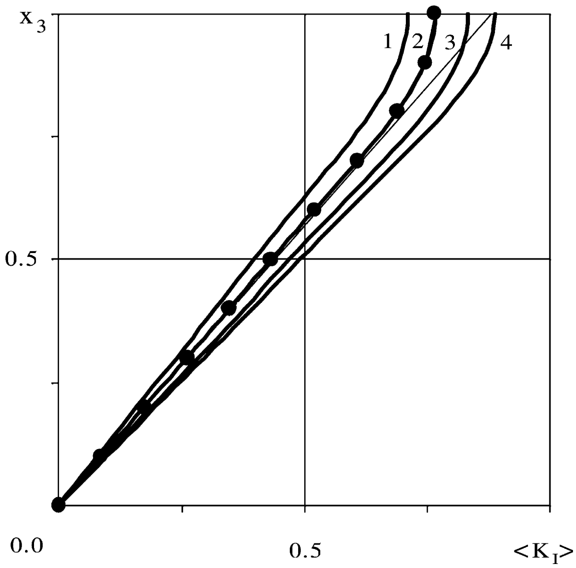

Fig. 4. Distribution of the relative stress intensity factor $\langle K_{\text{I}}\rangle$ for a rectilinear crack in bending.

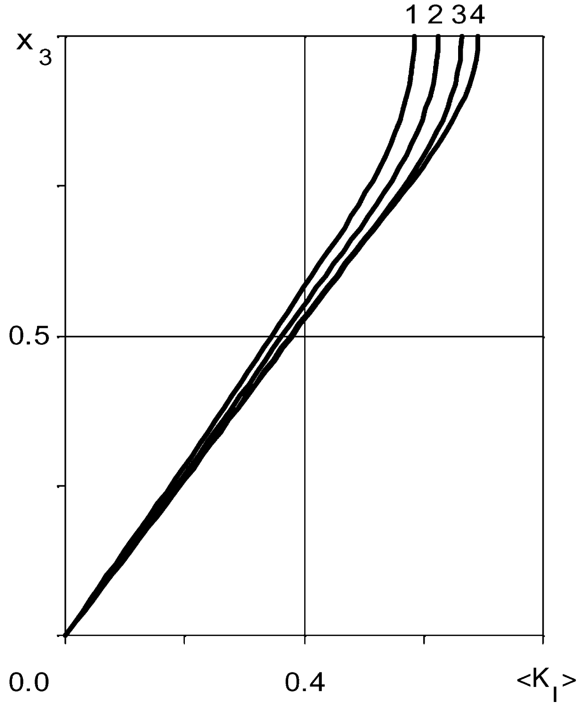

Fig. 5. Distribution of the relative stress intensity factor $\langle K_{\mathrm{I}} \rangle$ for a parabolic crack in bending.

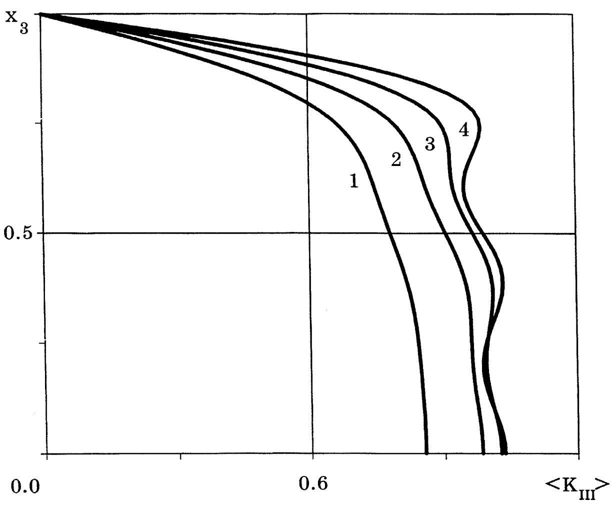

Fig. 6. Distribution of the relative stress intensity factor $\langle K_{\mathrm{III}} \rangle$ along the thickness coordinate x_3 for a rectilinear crack.

"thickness" coordinate are given in Figs. 6 and 7. The curves 1, 2, 3, and 4 (Fig. 6) were given for a straight crack ($p_1 = 1$ and $p_2 = 0$) for $h/l = 0.5$; 1; 2; and 4, respectively. The curves 1, 2, 3, and 4 in Fig. 7 were given for a parabolic crack ($p_1 = 1$ and $p_2 = 0.5$) for $h/l = 0.5$; 1; 2; and 4, respectively.

All the numerical results were obtained for the value of Poisson's ratio $v = 0.3$.

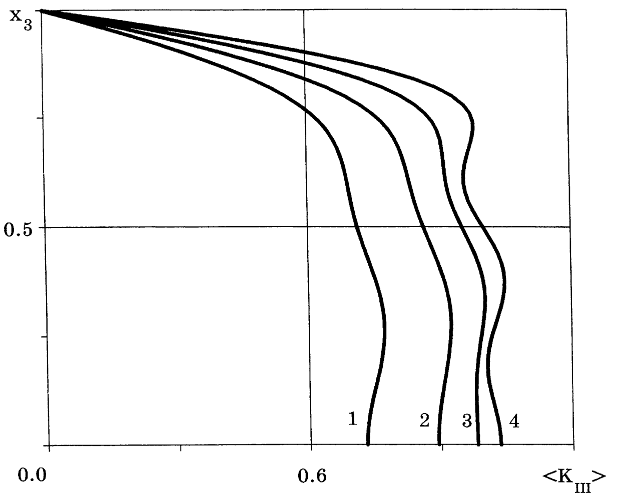

Fig. 7. Distribution of the relative stress intensity factor $\langle K_{III} \rangle$ along the thickness coordinate x_3 for a parabolic crack.

4. Conclusion

A novel method has been developed and demonstrated for the stress analysis of the boundary value problem of elastic layer weakened by through-thickness flaws. Several conclusions may be drawn about this method:

(1) The use of homogeneous solutions reduces a three-dimensional boundary value problem for a layer to a denumerable set of two-dimensional boundary value problems for metaharmonic functions.
(2) This approach enables an efficient solution of the problem of the correspondence of boundary conditions for the stress vector on the flaw surface to boundary conditions for each metaharmonic function.
(3) By virtue of rapid convergence of the developed algorithms, it is sufficient to reduce the infinite system of one-dimensional integral equations to a finite system with a quite small number of equations. The latter practically decreases the problem dimensionality by two units. In this sense, the new approach differs favorably from such well-known methods as the FEM and boundary element method.
(4) The disadvantages of this procedure lie in the awkwardness of analytical techniques used to obtain the system of one-dimensional integral equations.

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